CS 5000: Lecture 40

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Outline

- Recursive and Recursively Enumerable Sets
- Index Sets and Rice’s Theorem

Recursive and R.E. Sets

Review: R.E. Sets

The set B is recursively enumerable (r.e.) if there is a partially computable function $g(x)$ such that $B = \{x \in N \mid g(x)\downarrow\}$.

Theorem 4.3 (Ch. 4)

If B is a recursive set, then B is r.e.

Theorem 4.3: Proof

If B is recursive, $P_B(x)$ is computable.

Let $h(x)$ be defined as follows:

$$h(x) = \begin{cases} 1 \text{ if } P_B(x) = 1 \\ \uparrow \text{ if } P_B(x) = 0. \end{cases}$$

Theorem 4.3: Proof

Consider the following program P:

$[\mathrm{A}]\ \mathrm{IF}\ \alpha\left(P_B\left(X\right)\right)\ \mathrm{GOTO\ A.}$

$\mathrm{Y} \leftarrow 1$

Theorem 4.3: Proof

$$\psi_P^{(1)}(x) = h(x).$$

$$B = \{x \in N \mid h(x)\downarrow\}.$$

Theorem 4.4 (Ch. 4)

The set B is recursive if and only if B and $\overline{B}$ are both r.e.

Theorem 4.4 (Ch. 4): Proof

1. If B is recursive, then both B and its complement are r.e.
2. If B and its complement are both r.e., then B is recursive.

Proof: Part 1

If B is recursive, then, by Theorem 4.1 (Ch. 4), so is $\overline{B}$. Then, by Theorem 4.3 (Ch. 4), both B and $\overline{B}$ are r.e.

Proof: Part 2

Let B and $\overline{B}$ be r.e. Then

$$B = \{x \in N \mid g(x)\downarrow\}$$
$$\overline{B} = \{x \in N \mid h(x)\downarrow\}$$

Let $\Psi_P^{(1)}(x) = g(x)$ and $\Psi_Q^{(1)}(x) = h(x)$.

Proof: Part 2

Consider the following program M :

$[\mathrm{A}]\ \mathrm{IF}\ STP^{(1)}(X, \#(P), T)\ \mathrm{GOTO\ C1}$

$\quad \mathrm{IF}\ STP^{(1)}(X, \#(Q), T)\ \mathrm{GOTO\ C2}$

$\quad \mathrm{T} \leftarrow \mathrm{T} + 1$

$\quad \mathrm{GOTO\ A}$

$[\mathrm{C1}]\ \mathrm{Y} \leftarrow 1$

GOTO TO E

$[\mathrm{C2}]\ \mathrm{Y} \leftarrow 0$

Definition

$$W_n = \{x \in N \mid \Phi(x, n)\downarrow\}.$$

Theorem 4.6, Ch. 4

A set B is r.e. if and only if there is an n for which $B = W_n$.

Theorem 4.6: Proof

Assume that B is r.e. Then there exists a partially computable function g such that $B = \{x \in N \mid g(x)\downarrow\}$. Let G be the program that computes $g(x)$. Let $n = \#(G)$. Then $B = \{x \in N \mid \Phi(x, n)\downarrow\}$.

Theorem 4.6 (Ch. 4)

- Theorem 4.6 is also called Enumeration Theorem.
- This name originates from the fact that the following sequence enumerates all r.e. sets:
  - $W_0, W_1, W_2, W_3, \ldots$

Definition

$$K = \{n \in N \mid n \in W_n\}.$$

$$n \in W_n \Leftrightarrow \Phi(n,n)\downarrow \Leftrightarrow HALT(n,n).$$

K is the set of all numbers n such that a program whose number is n halts on n.

Theorem 4.7 (Ch. 4)

K is r.e. but not recursive.

Recall: Theorem 3.1 (Ch. 4)

$\Phi^{(n)}(x_1,\ldots,x_n,y) = \Psi_P^{(n)}(x_1,\ldots,x_n)$, where $y = \#(P)$.

For each $n > 0$, the function $\Phi^{(n)}(x_1,\ldots,x_n,y)$ is partially computable.

Theorem 4.7: Proof

By the Universality Theorem, $\Phi(n,n)$ is partially computable. Thus, since $K = \{n \in N \mid \Phi(n,n)\downarrow\}$, K is r.e.

Theorem 4.7: Proof

Assume that K is recursive. Then $\overline{K}$ is r.e. too. Thus, $\overline{K} = W_i$, for some i.

$$i \in \overline{K} \Leftrightarrow i \in W_i \Leftrightarrow i \in K.$$

Index Sets and Rice’s Theorem

Definition: Index Set

Γ is a collection of p.c. functions of one variable.

$R_\Gamma = \{t \in N \mid \Phi_t \in \Gamma\}.$

Index Sets: Examples

- The set of computable functions.
- The set of primitive recursive functions.
- The set of partially computable functions that are defined for all but a finite number of values.
- The set of viruses.
- The set of trigonometric functions.
- The set of derivatives of trigonometric functions.

Properties of Programs

R_Γ is recursive just in case there is an algorithm that accepts programs as input and returns TRUE or FALSE depending on whether a particular program is in Γ.

Rice's Theorem

Let Γ be a collection of partially computable functions of one variable. Let $f(x), g(x)$ be partially computable functions such that $f(x) \in \Gamma, g(x) \notin \Gamma$. Then R_Γ is not recursive.

Rice's Theorem: Original Formulation

Let C be a collection of p.c. functions of one variable. Then $\{x \mid \Phi_x \in C\}$ is recursive if and only if $C = \varnothing$ or C includes all partially computable functions of one variable.

Two Interesting Corollaries

- It is impossible to write a program that detects all viruses.
- It is impossible to write a program that detects all cases of plagiarism.