CS 5000: Lecture 39

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Outline

- Normal Form for (Partially) Computable Functions
- Sets and Characteristic Functions
- Recursively Enumerable Sets

Review: Theorem 3.3 (Ch. 4): Normal Form Theorem

Let $f(x_1,\ldots,x_n)$ be a partially computable function.
Then there is a primitive recursive predicate $R(x_1,\ldots,x_n,y_0,z)$
such that
Let $f(x_1,\ldots,x_n) = l\left(\min_z R(x_1,\ldots,x_n,y_0,z)\right)$, where y_0
is the number of a program that computes $f(x_1,\ldots,x_n)$.

Review: Theorem 3.4 (Ch. 4)

A function is partially computable if and only if it can be obtained from the initial functions by a finite number of applications of composition, recursion, and minimalization.

Proper Minimalization

$\min_{z} R(x_1, \ldots, x_n, y_0, z)$ can be a total function.

When does this happen?

This happens when for each $x_1, \ldots, x_n$ there
is at least one z such that $R(x_1, \ldots, x_n, y_0, z) = 1$.
If this is the case, $\min_{z} R(x_1, \ldots, x_n, y_0, z)$ is proper
minimalization.

Theorem 3.5 (Ch. 4)

A function is computable if and only if it can be obtained by a finite number of applications of composition, recursion, and proper minimalization.

Characteristic Functions

$B \subseteq N^m, m \geq 1.$ Then $P_B(x_1,\ldots,x_m)$ is a characteristic function of B if

$$P_B(x_1,\ldots,x_m) = \begin{cases} 1 \text{ if } (x_1,\ldots,x_m) \in B \\ 0 \text{ otherwise.} \end{cases}$$

Sets and Characteristic Functions

Let $B \subseteq N^m, m \geq 1.$ B belongs to some class of functions if and only if its characteristic function $P(x_1, \ldots, x_m)$ belongs to that class of functions.

A Note on Terminology

- F is a characteristic function of B.
- F computes B.
- The two above statements are synonymous.

Example 1

$$B = \{(x, y) \in N^2 \mid HALT(x, y)\}.$$

Then computing B is as hard as computig $HALT(x, y)$.

Theorem 4.1 (Ch. 4)

Let the sets B, C belong to some PRC class K. Then so do the sets $B \cup C$, $B \cap C$, $\overline{B}$.

Theorem 4.1: Proof

1) By Theorem 5.1 of Ch. 3, if K is a PRC class and P and Q belong to K, then so do $\overline{P}, P \vee Q$, and $P \& Q$.

2) Let $P_B(x_1,...,x_m)$ be the characteristic function of B.

3) Let $P_C(x_1,...,x_m)$ be the characteristic function of C.

4) Since B and C are in K, P_B and P_C are in K.

5) By Theorem 5.1 of Ch. 3, $\overline{P_B}, P_B \vee P_C, P_B \& P_C$ are in K.

Theorem 4.2, Ch. 4

Let K be a PRC class, and let B be a subset of $N^m, m \geq 1$. Then B belongs to K if and only if

$$B' = \{[x_1, \ldots, x_m] \in N \mid (x_1, \ldots, x_m) \in B\}$$

belongs to K.

Theorem 4.2: Proof

Let $P_B(x_1,...,x_m)$ be the characteristic function of B. Define $P_{B'}$ as follows:

$$P_{B'}(x) \Leftrightarrow P_B((x)_1,...,(x)_m) \,\&\, Lt(x) = m$$

Thus, if $P_B \in K$, so is $P_{B'}$.

Theorem 4.2: Proof

On the other hand, if $P_{B'}$ is the characteristic function of B', define P_B as follows:

$$P_B(x_1,...,x_m) \Leftrightarrow P_{B'}([x_1,...,x_m]).$$

If $P_{B'} \in K$, then $P_B \in K$.

Corollary 4.2

Let $B = \{[x, y] \in N \mid HALT(x, y)\}$. Then B is not computable.

Theorem 4.2 (Ch. 4) and N^m

- Theorem 4.2 (Ch. 4) essentially says that we do not need multi-dimensional spaces when we talk about computability on natural numbers.
- As long as we have functions $()_i$ and [ ], we can stay within N.
- [ ] is the primitive recursive function that computes Godel numbers.

Recursively Enumerable (R.E.) Sets: Definition

The set B is recursively enumerable (r.e.) if there is a partially computable function $g(x)$ such that $B = \{x \in N \mid g(x)\downarrow\}$.

Observations

- A set is r.e. when it is the domain of a partially computable function.
- Let *P* be a program that computes some partially computable function. Then all inputs to *P* on which it halts form an r.e. set.

Recommended Reading

- Section 4.4.