CS 5000: Lecture 26

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Outline

- Minimalization

Review: Theorem 6.1 (Ch. 3)

Let C be a PRC class. If $f(x_1,\ldots,x_n) \in C$,

then so do the functions

$$g(y, x_1,\ldots,x_n) = \sum_{t=0}^{y} f(t, x_1,\ldots,x_n)$$

and

$$h(y, x_1,\ldots,x_n) = \prod_{t=0}^{y} f(t, x_1,\ldots,x_n).$$

Corollary 6.2 (Ch. 3)

If $f(t, x_1, \ldots, x_n) \in C$ and C is PRC, then so do the functions

$$g(y, x_1, \ldots, x_n) = \sum_{t=1}^{y} f(t, x_1, \ldots, x_n)$$

$$h(y, x_1, \ldots, x_n) = \prod_{t=1}^{y} f(t, x_1, \ldots, x_n).$$

Review: Strict Bounded Quantification

$$(\forall t)_{<y} P(t, x_1, ..., x_n) \Leftrightarrow$$
$$(\forall t)_{\leq y} \left[t = y \vee P(t, x_1, ..., x_n) \right]$$

$$\left(\exists t\right)_{<y} P(t, x_1, ..., x_n) \Leftrightarrow$$
$$(\exists t)_{\leq y} \left[\alpha(t = y) \,\&\, P(t, x_1, ..., x_n) \right]$$

Theorem 6.3 (Ch. 3)

If the predicate $P(t, x_1, ..., x_n)$ belongs to some PRC class C, then so do the predicates

$(\forall t)_{\leq y} P(t, x_1, ..., x_n)$

and

$(\exists t)_{t \leq y} P(t, x_1, ..., x_n)$.

Proof 6.3

$$(\forall t)_{\leq y} P(t, x_1, \ldots, x_n) \Leftrightarrow$$

$$\left[\prod_{t=0}^{y} P(t, x_1, \ldots, x_n) \right] = 1$$

Proof 6.3

$$(\exists t)_{\leq y} P(t, x_1, \ldots, x_n) \Leftrightarrow$$

$$\alpha\left(\left[\sum_{t=0}^{y} P(t, x_1, \ldots, x_n)\right] = 0\right)$$

Example 1

Show that $y \mid x$ is primitive recursive.

Example 1

$$y \mid x \Leftrightarrow$$

$$(\exists t)_{\leq x}\left[y \cdot t = x\right]$$

Example 2

Show that Prime(x) is primitive recursive.

Example 2

A number is prime if it is greater than 1 and it has no divisors other than 1 and itself.

Prime(x) $\Leftrightarrow$

$$x > 1 \& (\forall t)_{\leq x}[t = 0 \vee t = 1 \vee t = x \vee \neg(t \mid x)].$$

Minimalization

0	1	2	3	4
P(0) = 0	**P(1) = 0**	**P(2) = 0**	**P(3) = 1**	**P(4) = ?**

$$\alpha(P(0)) = 1$$

$$\alpha(P(0)) \cdot \alpha(P(1)) = 1 \cdot 1 = 1$$

$$\alpha(P(0)) \cdot \alpha(P(1)) \cdot \alpha(P(2)) = 1 \cdot 1 \cdot 1 = 1$$

$$\alpha(P(0)) \cdot \ldots \cdot \alpha(P(n)) = 0, n \geq 3.$$

Minimalization

Suppose:

$$1. \left(\forall t\right)_{t<t_0} P(t, x_1, ..., x_n) = 0.$$

$$2. P(t_0, x_1, ..., x_n) = 1.$$

Then

$$\prod_{t=0}^{u} \alpha(P(t, x_1, ..., x_n)) = \begin{cases} 1 \text{ if } u < t_0 \\ 0 \text{ if } u \geq t_0. \end{cases}$$

Minimalization

Let $P(t, x_1, ..., x_n)$ be a predicate in some PRC class C. Let t_0 be the smallest value for which $P(t, x_1, ..., x_n) = 1$.

Consider

$$g(y, x_1, ..., x_n) = \sum_{u=0}^{y} \prod_{t=0}^{u} \alpha(P(t, x_1, ..., x_n)).$$

Example 3

Suppose $t_0 = 3, y = 4.$

$$g(4, x_1,..., x_n) = \sum_{u=0}^{4} \prod_{t=0}^{u} \alpha(P(t, x_1,..., x_n)) =$$

$$\prod_{t=0}^{0} \alpha(P(t, x_1,..., x_n)) + \prod_{t=0}^{1} \alpha(P(t, x_1,..., x_n)) +$$

$$\prod_{t=0}^{2} \alpha(P(t, x_1,..., x_n)) + \prod_{t=0}^{3} \alpha(P(t, x_1,..., x_n)) +$$

$$\prod_{t=0}^{4} \alpha(P(t, x_1,..., x_n)) = 1 + 1 + 1 + 0 + 0 = 3.$$

Example 4

Suppose $t_0 = 3, y = 1.$

$$g(1, x_1, ..., x_n) = \sum_{u=0}^{1} \prod_{t=0}^{u} \alpha(P(t, x_1, ..., x_n)) =$$

$$\prod_{t=0}^{0} \alpha(P(t, x_1, ..., x_n)) + \prod_{t=0}^{1} \alpha(P(t, x_1, ..., x_n))$$

$$= 1 + 1 = 2.$$

Lemma

$$(\forall y)_{<t_0}\, g(y, x_1, \ldots, x_n) = y + 1.$$

Proof

Let $y < t_0$. Then for every u in the range $[0, y]$, $\prod_{t=0}^{u} \alpha(P(t, x_1, ..., x_n)) = 1$. Since the range $[0, y]$ has $y+1$ elements,

$$\sum_{u=0}^{y} \prod_{t=0}^{u} \alpha(P(t, x_1, ..., x_n)) = y + 1.$$

Minimalization

If $t_0 \leq y$, then $g(y, x_1, ..., x_n) = \sum_{u<t_0} 1 = t_0$.

$g(y, x_1, ..., x_n)$ is the least value of t for which $P(t, x_1, ..., x_n)$ is true.

Minimalization: Definition

$\min_{t \leq y} P(t, x_1, \ldots, x_n)$ is the least value of t for which $P(t, x_1, \ldots, x_n)$ is true, if such a value exists. Otherwise, it is 0.

Minimalization: Definition

$$\min_{t \le y} P(t, x_1, \ldots, x_n) = \begin{cases} g(y, x_1, \ldots, x_n) \text{ if } (\exists t)_{\le y} P(t, x_1, \ldots, x_n) \\ 0 \qquad\qquad\qquad\qquad \text{otherwise.} \end{cases}$$

Theorem 7.1 (Ch. 3)

If $P(t, x_1, ..., x_n)$ belongs to some PRC class C and $f(y, x_1, ..., x_n) = \min_{t \le y} P(t, x_1, ..., x_n)$,

then $f(y, x_1, ..., x_n)$ also belongs to C.

Review: Theorem 5.4

Let C be a PRC class. Let the functions g, h and the predicate P belong to C.

Let

$$f(x_1,\ldots,x_n) = \begin{cases} g(x_1,\ldots,x_n) & \text{if } P(x_1,\ldots,x_n) \\ h(x_1,\ldots,x_n) & \text{otherwise.} \end{cases}$$

Then $f(x_1,\ldots,x_n)$ is in C.

Review: Theorem 6.3

If the predicate $P(t, x_1, ..., x_n)$ belongs to some PRC class C, then so do the predicates

$$(\forall t)_{\leq y} P(t, x_1, ..., x_n)$$

and

$$(\exists t)_{t \leq y} P(t, x_1, ..., x_n).$$

Theorem 7.1: Proof

$g(y, x_1, \ldots, x_n) \in C$, because

$\sum, \prod$, and α are in C. By Theorem 6.3,

$(\exists t)_{\leq y} P(t, x_1, \ldots, x_n)$ is also in C.

By Theorem 5.4, $\min_{t \leq y} P(t, x_1, \ldots, x_n)$ is

in C.

Example 5

Show that $\left\lfloor \frac{x}{y} \right\rfloor$ is primitive recursive.

Example 5

$$\left\lfloor \frac{x}{y} \right\rfloor \Leftrightarrow \min_{t \leq y}\left[(t+1) \cdot y > x\right]$$

Example 5

$$\left\lfloor \frac{7}{2} \right\rfloor \Leftrightarrow \min_{t \le y}\left[(t+1)\cdot 2 > 7\right] = 3,$$

$$\text{because } \left(3+1\right)\cdot 2 > 7.$$

Suggested Reading

- Sections 3.6, 3.7.