On Covering Problems of Rado

Sergey Bereg
University of Texas at Dallas

Adrian Dumitrescu
University of Wisconsin - Milwaukee

Minghui Jiang
Utah State University

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A Conjecture by T. Rado in 1928

For any set $\mathcal{S}$ of axis-parallel squares in the plane, there is always a subset $\mathcal{I}$ of pairwise disjoint squares such that

total area of squares in $\mathcal{I}$

is at least $1/4$ of

union area of squares in $\mathcal{S}$

More Precisely...

For a finite set $\mathcal{S}$ of convex sets in $\mathbb{R}^d$, the *union volume* (or *union area* when $d = 2$) of $\mathcal{S}$ is

$$|\mathcal{S}| = |\cup_{C \in \mathcal{S}} C|.$$

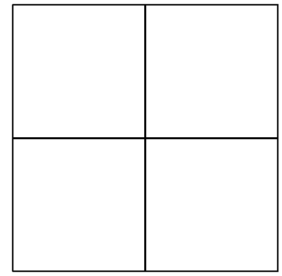
For a convex set S in $\mathbb{R}^d$, define

$$F(S) = \inf_{\mathcal{S}} \sup_{\mathcal{I}} \frac{|\mathcal{I}|}{|\mathcal{S}|},$$

where $\mathcal{S}$ ranges over all finite sets of convex sets in $\mathbb{R}^d$ that are homothetic to S, and $\mathcal{I}$ ranges over all independent subsets of $\mathcal{S}$.

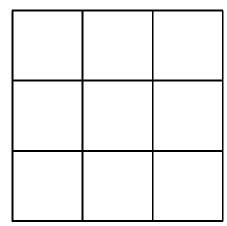
Is it true that $F(S) \geq 1/4$ for a square S?

Upper Bound



$F(S) \leq 1/4$

Lower Bound



$F(S) \geq 1/9$

Greedy algorithm: largest first (T. Rado 1928)

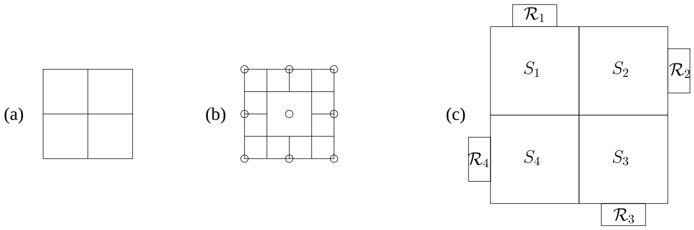
$F(S) \geq 1/8.6$

Semi-greedy algorithm: largest or neighbors (Zalgaller 1960)

The Question

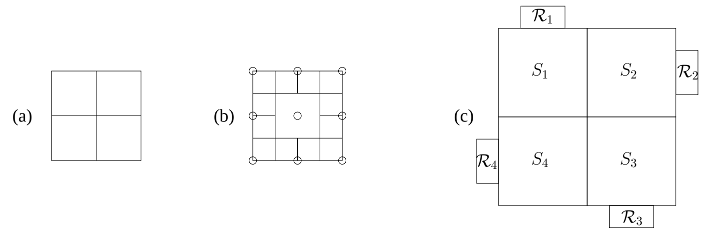
$$F(S) \stackrel{?}{=} 1/4$$

Disproved by Ajtai in 1973



(a) A system of 4 squares.
(b) An ambiguous system of 13 squares.
(c) Ajtai's construction shown schematically: $F(S) < 1/4$.

Improve Ajtai's System



Step 1: design a better gadget $\mathcal{R}$

Step 2: repeat the system in a tiling

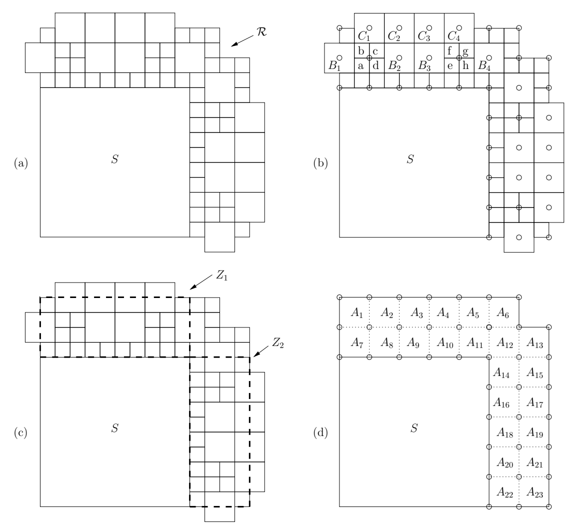
Our Results

For a square S,

$$1/8.6 < 1/8.4797\ldots \leq F(S) \leq 1/4.0421\ldots < 1/4.$$

Improved $F(S)$ bounds for disks, centrally symmetric convex sets in $\mathbb{R}^2$, arbitrary convex sets in $\mathbb{R}^2$, and hypercubes in $\mathbb{R}^d$.

Improved $f(S)$ bounds for *congruent* convex sets in $\mathbb{R}^2$.



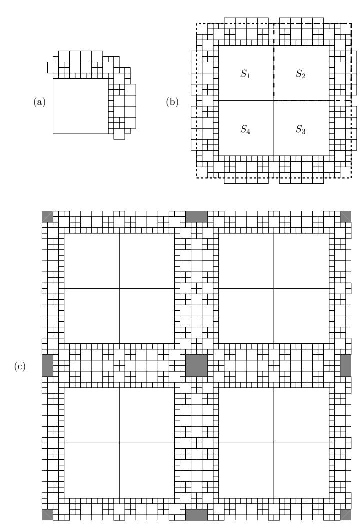
Improve Upper Bound

Estimate of Ajtai 1973:

$$1/4 - 1/1728 = 1/4.0092\ldots$$

Our result:

$$1/4 - 1/384 = 1/4.0421\ldots$$



Improve Lower Bound

T. Rado 1928 (largest first): $1/9$
R. Rado 1949 (largest or two neighbors): $1/8.75$
Zalgaller 1960 (largest or at most four neighbors): $1/8.6$

Our results:

largest or two neighbors
or two neighbors of a neighbor
or two neighbors of a neighbor of a neighbor...

$$1/8.5699\ldots$$

largest or three large neighbors

$$1/8.4797\ldots$$

Algorithm A1

- $\mathcal{S} = \{S_1, \ldots, S_n\}$ is a set of axis-parallel squares.
- T_i is the smallest axis-parallel square that contains all squares in $\mathcal{S}$ that intersect S_i.
- x_i is the side length of S_i.
- y_i is the side length of T_i.
- $z_i = y_i - x_i$.
- λ is a constant (will be set to $8.5699\ldots$).

To construct an independent set $\mathcal{I}$, initialize $\mathcal{I}$ to be empty, then repeat the following *selection round* until $\mathcal{S}$ is empty.

Selection Round of Algorithm A1

1. Find the largest square S_l in $\mathcal{S}$. Assume without loss of generality that $x_l = 1$.
2. If $y_l \leq \sqrt{\lambda}$, add S_l to $\mathcal{I}$, delete from $\mathcal{S}$ the squares that intersect S_l, then return. Otherwise, set $k \leftarrow l$ and continue with the next step.
3. Let S_i and S_j be two squares that intersect S_k and touch two opposite sides of T_k. If both z_i and z_j are at most z_k, add S_i and S_j to $\mathcal{I}$, delete from $\mathcal{S}$ the squares that intersect S_i or S_j, then return. Otherwise, set $k \leftarrow i$ or j such that z_k increases, then repeat this step.

The idea: keep searching if a neighbor has "more" neighbors.

In Every Iteration of Step 3

Since S_i and S_j are two squares that intersect S_k and touch two opposite sides of T_k,

$$x_i + x_j + x_k \geq y_k = x_k + z_k.$$

Then,

$$x_i + x_j \geq z_k.$$

So when we reach the last iteration of Step 3...

$$\begin{aligned}
\frac{|T_i \cup T_j|}{|S_i| + |S_j|} &= \frac{|T_i| + |T_j| - |T_i \cap T_j|}{|S_i| + |S_j|} \\
&\leq \frac{(x_i + z_k)^2 + (x_j + z_k)^2}{x_i^2 + x_j^2} - \frac{x_k^2}{x_i^2 + x_j^2} \\
&= 1 + \frac{2z_k(x_i + x_j) + 2z_k^2}{x_i^2 + x_j^2} - \frac{x_k^2}{x_i^2 + x_j^2} \\
&\leq 1 + \frac{2z_k(x_i + x_j) + 2z_k^2}{(x_i + x_j)^2/2} - \frac{(\sqrt{\lambda} - 2)^2}{2} \\
&= 1 + \frac{4z_k}{x_i + x_j} + \frac{4z_k^2}{(x_i + x_j)^2} - \frac{(\sqrt{\lambda} - 2)^2}{2} \\
&\leq 9 - (\sqrt{\lambda} - 2)^2/2
\end{aligned}$$

Two Cases

1. If S_l is selected (in Step 2),
$$\frac{|T_l|}{|S_l|} \leq \lambda.$$
2. If S_i and S_j are selected (in the last iteration of Step 3),
$$\frac{|T_i \cup T_j|}{|S_i| + |S_j|} \leq 9 - (\sqrt{\lambda} - 2)^2/2.$$

$$9 - (\sqrt{\lambda} - 2)^2/2 = \lambda$$

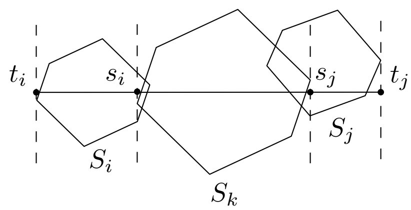
$$\lambda = (\sqrt{46} + 2)^2/9 = 8.5699\ldots$$

$F(S) \geq 1/\lambda$ for a square S.

More Lower Bounds

$$3^d - (\lambda_d^{1/d} - 2)^d/2 = \lambda_d$$

$F(S) \geq 1/\lambda_d$ for a hypercube S in $\mathbb{R}^d$.



$F(S) \geq 1/8.5699\ldots$ for a centrally symmetric convex set in $\mathbb{R}^2$.
$F(S) \geq 1/8.3539\ldots$ for a disk (by a tighter analysis).

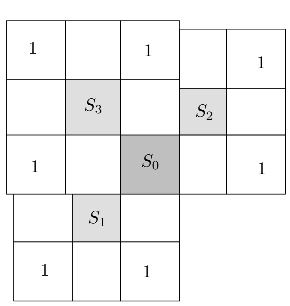
Selection Round of Algorithm A2

1. Let S_0 be the largest square in $\mathcal{S}$.
   Assume without loss of generality that S_0 is a unit square.
   Let $\mathcal{S}_0 \subseteq \mathcal{S} \setminus \{S_0\}$ be the set of squares of side length at least $s = 0.8601\ldots$ that intersect S_0.
2. If $\mathcal{S}_0$ contains three disjoint squares S_1, S_2, and S_3, then add S_1, S_2, and S_3 to $\mathcal{I}$. Otherwise add S_0 to $\mathcal{I}$.
3. For each square S_i added to $\mathcal{I}$, remove from $\mathcal{S}$ the squares that intersect S_i.

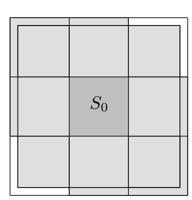
The idea: take the largest or three *large* neighbors.

Two Cases

(a)



(b)



$$\frac{8 + 3s^2 + 10s}{3s^2} = 7 + 2s^2 = 8.4797\ldots$$

Summary

For a square S,

$$1/8.6 < 1/8.4797\ldots \leq F(S) \leq 1/4.0421\ldots < 1/4.$$

Improved $F(S)$ bounds for disks, centrally symmetric convex sets in $\mathbb{R}^2$, arbitrary convex sets in $\mathbb{R}^2$, and hypercubes in $\mathbb{R}^d$.

Improved $f(S)$ bounds for *congruent* convex sets in $\mathbb{R}^2$.

Thank you!