

On Covering Problems of Rado

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A Conjecture by T. Rado in 1928

For any set \mathcal{S} of axis-parallel squares in the plane, there is always a subset \mathcal{I} of pairwise disjoint squares such that

total area of squares in \mathcal{I}

is at least $1/4$ of

union area of squares in \mathcal{S}

More Precisely...

For a finite set \mathcal{S} of convex sets in \mathbb{R}^d , the *union volume* (or *union area* when $d = 2$) of \mathcal{S} is

$$|\mathcal{S}| = |\cup_{C \in \mathcal{S}} C|.$$

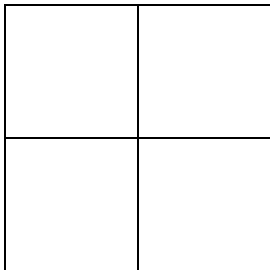
For a convex set S in \mathbb{R}^d , define

$$F(S) = \inf_S \sup_{\mathcal{I}} \frac{|\mathcal{I}|}{|S|},$$

where \mathcal{S} ranges over all finite sets of convex sets in \mathbb{R}^d that are homothetic to S , and \mathcal{I} ranges over all independent subsets of \mathcal{S} .

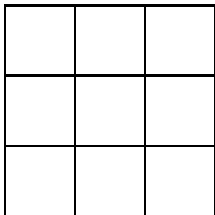
Is it true that $F(S) \geq 1/4$ for a square S ?

Upper Bound



$$F(S) \leq 1/4$$

Lower Bound



$$F(S) \geq 1/9$$

Greedy algorithm: largest first (T. Rado 1928)

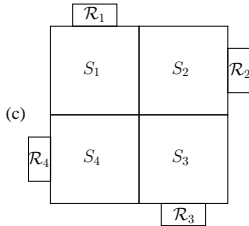
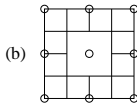
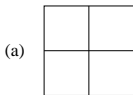
$$F(S) \geq 1/8.6$$

Semi-greedy algorithm: largest or neighbors (Zalgaller 1960)

The Question

$$F(S) \stackrel{?}{=} 1/4$$

Disproved by Ajtai in 1973



(a) A system of 4 squares.

(b) An ambiguous system of 13 squares.

(c) Ajtai's construction shown schematically: $F(S) < 1/4$.

Our Results

For a square S ,

$$1/8.6 < 1/8.4797\dots \leq F(S) \leq 1/4.0421\dots < 1/4.$$

Improved $F(S)$ bounds for disks, centrally symmetric convex sets in \mathbb{R}^2 , arbitrary convex sets in \mathbb{R}^2 , and hypercubes in \mathbb{R}^d .

Improved $f(S)$ bounds for *congruent* convex sets in \mathbb{R}^2 .

Improve Upper Bound

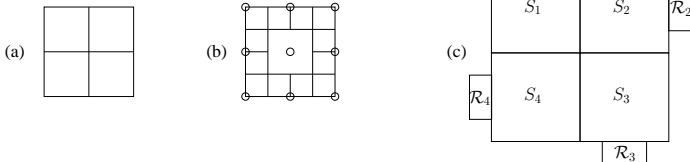
Estimate of Ajtai 1973:

$$1/4 - 1/1728 = 1/4.0092 \dots$$

Our result:

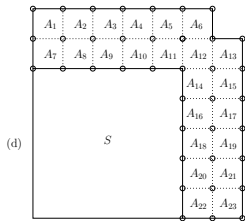
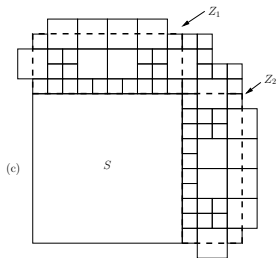
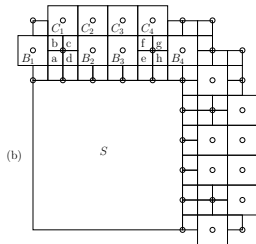
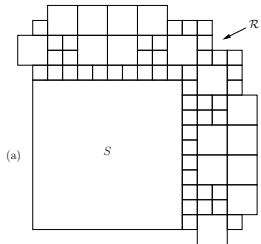
$$1/4 - 1/384 = 1/4.0421 \dots$$

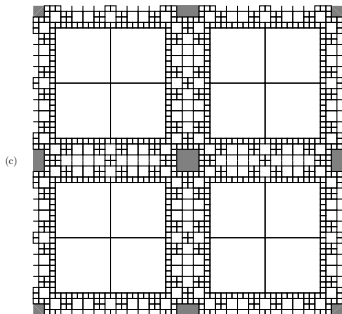
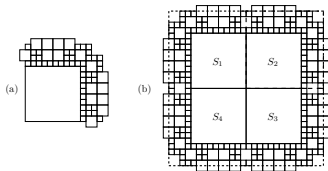
Improve Ajtai's System



Step 1: design a better gadget \mathcal{R}

Step 2: repeat the system in a tiling





Improve Lower Bound

T. Rado 1928 (largest first): $1/9$

R. Rado 1949 (largest or two neighbors): $1/8.75$

Zalgaller 1960 (largest or at most four neighbors): $1/8.6$

Our results:

largest or two neighbors

or two neighbors of a neighbor

or two neighbors of a neighbor of a neighbor...

$1/8.5699\dots$

largest or three large neighbors

$1/8.4797\dots$

Algorithm A1

- $\mathcal{S} = \{S_1, \dots, S_n\}$ is a set of axis-parallel squares.
- T_i is the smallest axis-parallel square that contains all squares in \mathcal{S} that intersect S_i .
- x_i is the side length of S_i .
- y_i is the side length of T_i .
- $z_i = y_i - x_i$.
- λ is a constant (will be set to 8.5699...).

To construct an independent set \mathcal{I} , initialize \mathcal{I} to be empty, then repeat the following *selection round* until \mathcal{S} is empty.

Selection Round of Algorithm A1

1. Find the largest square S_l in \mathcal{S} .
Assume without loss of generality that $x_l = 1$.
2. If $y_l \leq \sqrt{\lambda}$, add S_l to \mathcal{I} , delete from \mathcal{S} the squares that intersect S_l , then return. Otherwise, set $k \leftarrow l$ and continue with the next step.
3. Let S_i and S_j be two squares that intersect S_k and touch two opposite sides of T_k . If both z_i and z_j are at most z_k , add S_i and S_j to \mathcal{I} , delete from \mathcal{S} the squares that intersect S_i or S_j , then return. Otherwise, set $k \leftarrow i$ or j such that z_k increases, then repeat this step.

The idea: keep searching if a neighbor has “more” neighbors.

In Every Iteration of Step 3

Since S_i and S_j are two squares that intersect S_k and touch two opposite sides of T_k ,

$$x_i + x_j + x_k \geq y_k = x_k + z_k.$$

Then,

$$x_i + x_j \geq z_k.$$

So when we reach the last iteration of Step 3...

$$\begin{aligned}
\frac{|T_i \cup T_j|}{|S_i| + |S_j|} &= \frac{|T_i| + |T_j| - |T_i \cap T_j|}{|S_i| + |S_j|} \\
&\leq \frac{(x_i + z_k)^2 + (x_j + z_k)^2}{x_i^2 + x_j^2} - \frac{x_k^2}{x_i^2 + x_j^2} \\
&= 1 + \frac{2z_k(x_i + x_j) + 2z_k^2}{x_i^2 + x_j^2} - \frac{x_k^2}{x_i^2 + x_j^2} \\
&\leq 1 + \frac{2z_k(x_i + x_j) + 2z_k^2}{(x_i + x_j)^2/2} - \frac{(\sqrt{\lambda} - 2)^2}{2} \\
&= 1 + \frac{4z_k}{x_i + x_j} + \frac{4z_k^2}{(x_i + x_j)^2} - \frac{(\sqrt{\lambda} - 2)^2}{2} \\
&\leq 9 - (\sqrt{\lambda} - 2)^2/2
\end{aligned}$$

Two Cases

1. If S_l is selected (in Step 2),

$$\frac{|T_l|}{|S_l|} \leq \lambda.$$

2. If S_i and S_j are selected (in the last iteration of Step 3),

$$\frac{|T_i \cup T_j|}{|S_i| + |S_j|} \leq 9 - (\sqrt{\lambda} - 2)^2/2.$$

$$9 - (\sqrt{\lambda} - 2)^2/2 = \lambda$$

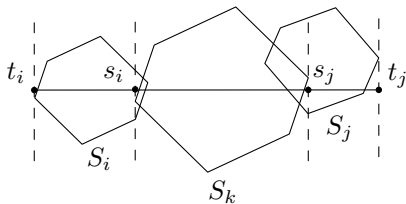
$$\lambda = (\sqrt{46} + 2)^2/9 = 8.5699\dots$$

$F(S) \geq 1/\lambda$ for a square S .

More Lower Bounds

$$3^d - (\lambda_d^{1/d} - 2)^d / 2 = \lambda_d$$

$F(S) \geq 1/\lambda_d$ for a hypercube S in \mathbb{R}^d .



$F(S) \geq 1/8.5699 \dots$ for a centrally symmetric convex set in \mathbb{R}^2 .

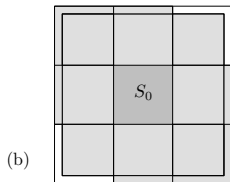
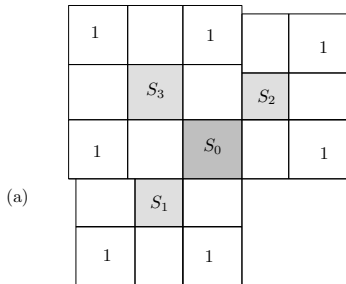
$F(S) \geq 1/8.3539 \dots$ for a disk (by a tighter analysis).

Selection Round of Algorithm A2

1. Let S_0 be the largest square in \mathcal{S} .
Assume without loss of generality that S_0 is a unit square.
Let $\mathcal{S}_0 \subseteq \mathcal{S} \setminus \{S_0\}$ be the set of squares of side length at least $s = 0.8601\dots$ that intersect S_0 .
2. If \mathcal{S}_0 contains three disjoint squares $S_1, S_2,$ and S_3 , then add $S_1, S_2,$ and S_3 to \mathcal{I} . Otherwise add S_0 to \mathcal{I} .
3. For each square S_i added to \mathcal{I} , remove from \mathcal{S} the squares that intersect S_i .

The idea: take the largest or three *large* neighbors.

Two Cases



$$\frac{8 + 3s^2 + 10s}{3s^2} = 7 + 2s^2 = 8.4797\dots$$

Summary

For a square S ,

$$1/8.6 < 1/8.4797 \dots \leq F(S) \leq 1/4.0421 \dots < 1/4.$$

Improved $F(S)$ bounds for disks, centrally symmetric convex sets in \mathbb{R}^2 , arbitrary convex sets in \mathbb{R}^2 , and hypercubes in \mathbb{R}^d .

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Thank you!