

Connect-the-dots in a Graph and Buffon's Needle on a Chessboard: Two Problems in Assisted Navigation

Minghui Jiang and Vladimir Kulyukin

Department of Computer Science, Utah State University

JCIS, July 21, 2007

Two Problems

With applications to assisted navigation:

- **Connect-the-dots in a graph** is a graph-theoretical problem with application to robot indoor localization.
- **Buffon's needle on a chessboard** is a problem in geometric probability with application to the design of RFID-enabled surface for robot-assisted navigation.

Smart Navigation Device

- Aging adults have increasing difficulties in everyday activities due to the **decline in their sensory-motor abilities**, and need smart robotic devices for assisted navigation.
- An important aspect of a navigation device is its ability to **determine its own location** when carried around by a user.

Numerous localization methods are available:

- Markov localization;
- RFID-based localization;
- Wireless localization.

Each method has its own vulnerability.

Wireless Localization

Kulyukin and Nicholson, RESNA 2005:

- An indoor environment contains many wireless access points.
- The aggregate wireless signals from the access points have different patterns at different locations.
- Machine learning algorithms classify the locations based on the received wireless signals.

The classifier can make mistakes when the environment is noisy.

A Reasonable Assumption

- A user equipped with a navigation device often has a limited speed of movement.
- The current location of the device is always "close" to its most recent location.

It is possible to improve the accuracy of a localization method by correcting certain mistakes that are apparent from

- the location history and
- the environment geometry.

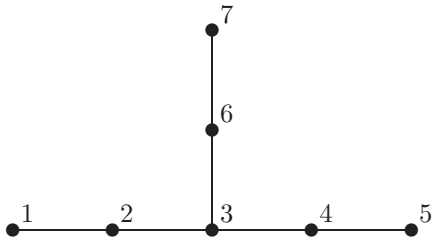
Environment Modeled as a Graph

- Vertices correspond to the discrete locations in the environment;
- Edges correspond to the paths, the stairs, and the corridors that connect the discrete locations.

In this graph, the assumption that the current location of the device is “close” to a recent location can be formulated as a constraint:

- two consecutive locations must correspond to either the same vertex or two adjacent vertices in the graph.

An Example



If a localization method reports the sequence of locations $\langle 1, 2, 7, 4, 5 \rangle$, then 7 is probably misclassified and may be corrected to 3.

Distance between Sequences

For a sequence S ,

- denote by $|S|$ the length of S ,
- denote by $S[i]$ the i 'th character of S ,
- denote by $S[i, j]$ as the subsequence $S[i] \dots S[j]$.

Define the *distance* between two sequences S and T of the same length l as

$$\text{dist}(S, T) = \sum_{i=1}^l d(S[i], T[i]),$$

where $d(u, v)$ is 0 if $u = v$ and is 1 otherwise.

Graph-Conforming Sequence

A sequence S *conforms* to a graph G if

1. every character of S is a vertex of G ;
2. every two consecutive characters of S are either the same vertex or two adjacent vertices of G .

Connect-the-Dots in a Graph

Given

- a connected graph $G = (V, E)$;
- a sequence S .

Find a sequence T of the same length as S such that

- T conforms to G ;
- the distance $\text{dist}(S, T)$ is minimized.

An Algorithm

We design a dynamic programming algorithm for the connect-the-dots problem.

For simplicity, we only demonstrate how to compute the minimum distance $\text{dist}(S, T)$; the actual sequence T that achieves the minimum distance with S can be reconstructed in the same running time using standard techniques in dynamic programming.

Dynamic Programming

Denote by $D[k][v]$ the minimum distance between the subsequence $S[1, k]$ and a graph-conforming sequence of the same length k ending at the vertex $v \in V$.

The base condition:

$$D[1][v] = d(S[1], v)$$

The recurrence:

$$D[k+1][v] = \min \left\{ \begin{array}{l} D[k][v] + d(S[k+1], v) \\ \min_{\{u,v\} \in E} \left\{ D[k][u] + d(S[k+1], v) \right\} \end{array} \right\}$$

The minimum distance is $\min_{v \in V} D[|S|][v]$.

The Analysis

Time:

- For each k , $O(|V| + |E|) = O(|E|)$;
- Overall, $O(|E||S|)$.

Space:

- $O(|V|)$ to compute the minimum distance only: the D table for at most two consecutive values of k ;
- $O(|V||S|)$ to reconstruct the graph-conforming sequence: a partial graph-conforming sequence ending at each vertex $v \in V$ as k increments from 1 to $|S|$.

The Main Result

Theorem 1 Given a connected graph $G = (V, E)$ a sequence S , there is an $O(|E||S|)$ time and $O(|V||S|)$ space algorithm that finds a sequence T of the same length as S such that T conforms to G and that $\text{dist}(S, T)$ is minimized.

Easy extensions:

- **Weighted case:** the distance $d(u, v)$ between two vertices u and v is not the Hamming distance but an arbitrary given weight;
- **On-line case:** the sequence S is given one character at a time.

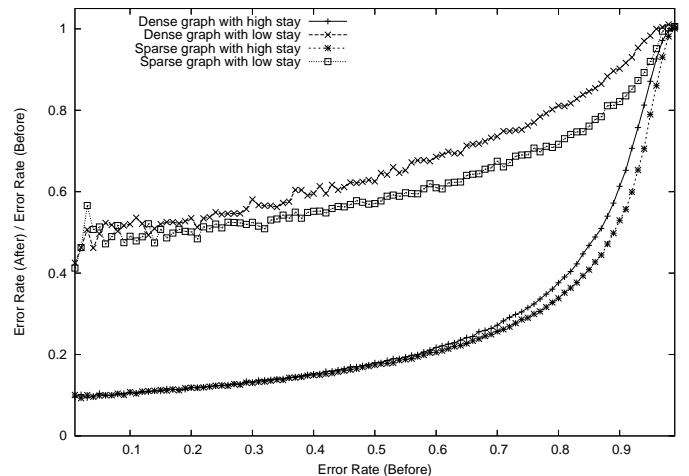
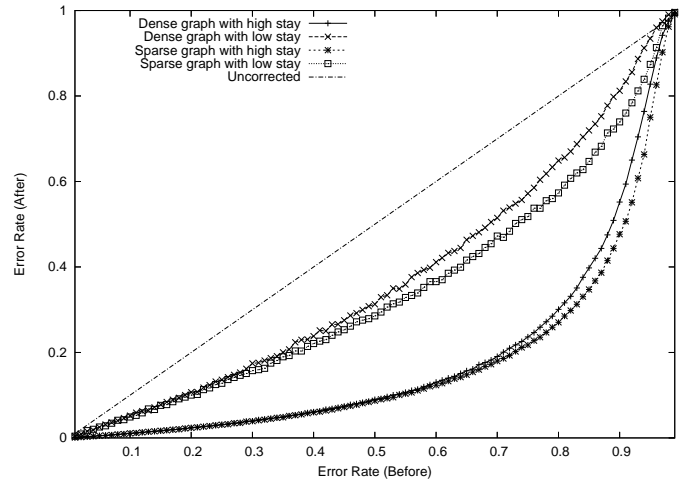
Experiments

Graph:

- Start with a 50×50 grid graph: each vertex (except the boundary) has 4 neighbors; number m of edges is about 2 times number n of vertices.
- Select random edges: dense degree $m = 1.8n$; sparse degree $m = 1.1n$.

Sequence:

- Random walk starting at the center of the graph: high stay probability 0.9; low stay probability 0.1.
- Error injection with ratio from 0.0 to 1.0.



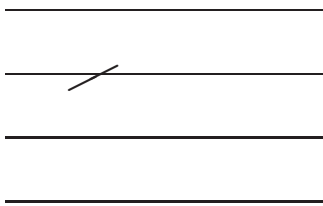
Buffon's Needle

- First posed by the French naturalist Buffon in 1733.
- Probably the best-known problem in the field of geometric probability.

“At Random” Means ...

y and θ are independently and uniformly distributed in the ranges $[0, h/2]$ and $[0, \pi/2]$, respectively.

Buffon's Needle on Parallel Lines



Buffon's Idea

Buffon considered the case that $l < h$:

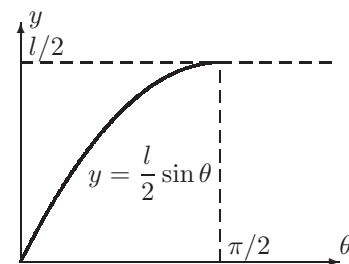
- The needle can intersect at most one line;
- The needle cuts one of the parallel lines if and only if $0 \leq y \leq (l/2) \sin \theta$.

A needle is dropped at random on the plane marked by equidistant parallel lines. What is the probability that the needle cuts (intersects) at least one line?

Four Parameters

- l is the length of the needle,
- h is the distance between two consecutive parallel lines.
- y is the minimum distance from the needle's midpoint to a line.
- θ is the minimum angle formed by the needle and the parallel lines;

The Solution



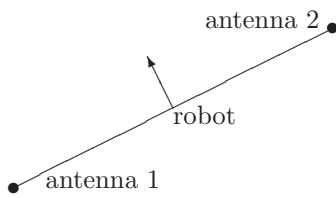
It is clear that $0 \leq y \leq h/2$ and $0 \leq \theta \leq \pi/2$.

$$\Pr\left(0 \leq y \leq (l/2) \sin \theta\right) = \frac{\int_0^{\pi/2} \int_0^{(l/2) \sin \theta} d\theta dy}{(\pi/2)(h/2)} = \frac{2l}{\pi h}.$$

RFID-Enabled Surface

- An RFID-enabled surface is a surface embedded with a set of passive radio frequency identification sensors.
- Each RFID sensor has an active area on the surface.
- When an antenna moves above the active area of a sensor, it receives a uniquely coded signal from the sensor and gets a *read*.

A Robot with Two Antenna

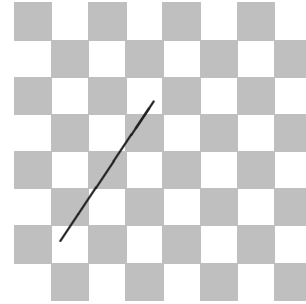


- When an antenna gets a read, the robot can look up the location of the RFID sensor from a table, which gives a rough estimate of the actual location of the antenna.
- When both antenna get reads, the robot can compute from the locations of the two antenna its own location and orientation.

Aggregate Active Area

- For the ease of implementation, the RFID sensors are often placed in a grid formation.
- Because of the electromagnetic interference between the sensors, the aggregate active area, above which the antenna get valid reads, manifests a pattern that resembles the black cells in a chessboard.
- The robot can compute its own location and orientation only when both antenna get reads, that is, when both antenna are in active area, or, when both antenna are in black cells of a chessboard.

Buffon's Needle on a Chessboard



A needle is dropped at random on a chessboard. What the probability that the needle's two endpoints are in two cells of the same color?

A Characterization

Let u and v , respectively, be the numbers of horizontal lines and vertical lines that the needle cuts.

Let $n = \lfloor l/h \rfloor$.

Clearly, both u and v are in the range $[0, n + 1]$.

Proposition 1 *The two endpoints of the needle are in two cells of the same color if and only if $u + v$ is even.*

The Proof

Let A and B be the two cells containing the two endpoints of the needle. Let C be a cell in the same column as A and in the same row as B .



- A and C have the same color if and only if the number u of horizontal grid lines between them is even;
- B and C have the same color if and only if the number v of vertical grid lines between them is even.

1. u and v are even: A , B , and C have the same color.
2. u and v are odd: A and B have the same color; C has a different color.
3. u is even, v is odd: A and C have the same color; B has a different color.
4. u is odd, v is even: B and C have the same color; A has a different color.

Please email your questions and comments to:

mjiang@cc.usu.edu

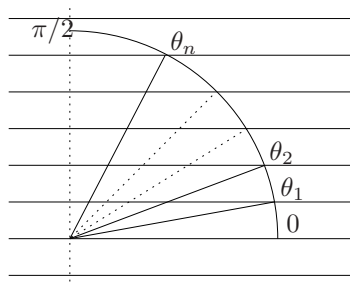
The Reduced Problem

The probability that the two endpoints of the needle are in two cells of the same color is

$$\Pr(\text{same color}) = \sum_{\substack{u,v \in [0,n+1] \\ u+v \text{ is even}}} \Pr(u,v), \quad (1)$$

where $\Pr(u,v)$ denotes the probability that the needle cuts exactly u horizontal lines and v vertical lines.

Compute $\Pr(u,v)$



Standard “sliding” technique with integrals. Omitted from this talk. Please refer to the paper for details.